

# ON THE $f$ -VECTORS OF GELFAND-CETLIN POLYTOPES

BYUNG HEE AN, YUNHYUNG CHO, AND JANG SOO KIM

**ABSTRACT.** A *Gelfand-Cetlin polytope* is a convex polytope obtained as an image of certain completely integrable system on a partial flag variety. In this paper, we give an equivalent description of the face structure of a GC-polytope in terms of so called *the face structure of a ladder diagram*. Using our description, we obtain a partial differential equation whose solution is the exponential generating function of  $f$ -vectors of GC-polytopes. This solves the open problem (2) posed by Gusev, Kritchenko, and Timorin in [GKT].

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## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let us fix a positive integer  $n$  and let  $\mathbf{n} = (n_0, n_1, \dots, n_r, n_{r+1})$  be a sequence of integers such that  $0 = n_0 < n_1 < n_2 < \dots < n_r < n_{r+1} = n$  for some  $r > 0$ . For a sequence  $\lambda = (\lambda_1, \dots, \lambda_n)$  of real numbers such that

$$\lambda_1 = \dots = \lambda_{n_1} > \lambda_{n_1+1} = \dots = \lambda_{n_2} > \dots > \lambda_{n_r+1} = \dots = \lambda_n,$$

the *Gelfand-Cetlin polytope*, or simply *the GC-polytope*, denoted by  $\mathcal{P}_\lambda$  is a convex polytope lying on  $\mathbb{R}^d$  ( $d = \frac{n(n-1)}{2}$ ) consisting of points  $(\lambda_i^{(j)})_{i,j} \in \mathbb{R}^d$  satisfying

$$\lambda_i^{(j+1)} \geq \lambda_i^{(j)} \geq \lambda_{i+1}^{(j)}, \quad 1 \leq i \leq n-1, \quad 1 \leq j \leq n-i$$

where  $\lambda_i^{(n-i+1)} := \lambda_i$  for all  $i = 1, \dots, n$ . Equivalently,  $(\lambda_i^{(j)})_{i,j} \in \mathcal{P}_\lambda$  if and only if it satisfies

$$(1.1) \quad \begin{array}{ccccccc} \lambda_1 & & \lambda_2 & & \lambda_3 & \cdots & \lambda_{n-1} & & \lambda_n \\ & \searrow & \nearrow & \searrow & \nearrow & \ddots & \searrow & \nearrow & \\ & & \lambda_1^{(n-1)} & & \lambda_2^{(n-2)} & & & & \lambda_{n-1}^{(1)} \\ & & \searrow & \nearrow & \ddots & & \searrow & \nearrow & \\ & & & \lambda_1^{(n-2)} & & & & \lambda_{n-2}^{(1)} & \\ & & & \searrow & \nearrow & & \searrow & \nearrow & \\ & & & & \ddots & & \ddots & & \\ & & & & & \searrow & \nearrow & & \\ & & & & & & \lambda_1^{(1)} & & \end{array}$$

for  $1 \leq i \leq n-1$  and  $1 \leq j \leq n-i$ .

The theory of GC-polytopes has been studied from various aspects, such as the representation theory of  $GL_n(\mathbb{C})$  ([GC], [GKT], [LMc]), and the geometry of Schubert varieties ([Ki], [Ko], [KM], [KST]). In the context of toric

*Date:* June 22, 2016.

The first and the second authors were supported by IBS-R003-D1. The third author was partially supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF- 2013R1A1A2061006).

geometry, GC-polytopes correspond to (very singular) projective toric varieties which can be regarded as toric degenerations of flag varieties. Thus to study of GC-polytopes in the sense of convex geometry is one of the natural way of understanding how to degenerate flag varieties to projective toric varieties, see [HK].

However, the combinatorics of GC-polytopes seems to be not quite well-understood. Recently, Gusev, Kiritchenko, and Timorin [GKT] studied the number of vertices of GC-polytopes. More precisely, they provided certain PDE system such that the solution is a power series with multi-variable  $x = (x_1, \dots, x_n)$  such that each coefficient of  $x^I$ , where  $I$  is an multi-index, is the number of vertices of the GC-polytope corresponding to  $I$ , see Section 1.4 for more details.

This paper concerns the enumerative combinatorics on Gelfand-Cetlin polytopes, in particular a counting the number faces in each dimension. Also, we provides the answer for the open question posed in [GKT, open problem (2) of page 968], see Theorem 1.12 and Remark 1.14.

**1.1. Geometric aspects of GC-polytopes.** A GC-polytope is closely related to the geometry of a partial flag variety, see [Ki], [Ko], [KM], and [NNU]. Even though we do not use the theory of GC-polytopes on the algebraic nor geometric aspects in this paper, we briefly explain a connection between GC-polytopes and the geometry of partial flag varieties as we see below.

A *partial flag variety*  $\mathcal{F}\ell(\mathbf{n})$  is an example of a projective Fano variety defined by

$$\mathcal{F}\ell(\mathbf{n}) = \{V_\bullet := 0 \subset V_1 \subset \dots \subset V_r \subset \mathbb{C}^n \mid \dim_{\mathbb{C}} V_i = n_i\}.$$

We can easily check that the linear  $U(n)$ -action on  $\mathbb{C}^n$  induces a transitive  $U(n)$ -action on  $\mathcal{F}\ell(\mathbf{n})$  with the stabilizer isomorphic to  $U(k_1) \times \dots \times U(k_{r+1})$  where  $k_i = n_i - n_{i-1}$  for  $i = 1, \dots, r+1$ . In other words,  $\mathcal{F}\ell(\mathbf{n})$  is diffeomorphic to a homogeneous space

$$\mathcal{F}\ell(\mathbf{n}) \cong U(n)/U(k_1) \times \dots \times U(k_{r+1}).$$

In the symplectic point of view,  $\mathcal{F}\ell(\mathbf{n})$  can be described as a co-adjoint orbit of  $U(n)$  as follows. Let  $U(n)$  be the set of  $n \times n$  unitary matrices and let  $\mathfrak{u}(n)$  be the Lie algebra of  $U(n)$ , which is the set of  $n \times n$  skew-hermitian matrices. Then we may identify the dual vector space  $\mathfrak{u}(n)^*$  with the set of  $n \times n$  hermitian matrices  $\mathcal{H} = i\mathfrak{u}(n)$  via the inner product

$$\langle X, Y \rangle = \text{tr}(XY)$$

on  $\mathcal{H}$  so that  $\mathfrak{u}(n)^*$  with the co-adjoint  $U(n)$ -action is  $U(n)$ -equivariantly diffeomorphic to  $\mathcal{H}$  with the conjugate action of  $U(n)$ , see [Au, page 51] for the detail.

Let  $I_\lambda$  be the diagonal matrix  $I_\lambda = \text{diag}(\lambda_1, \dots, \lambda_n) \in \mathcal{H}$ . Then the orbit of  $I_\lambda$  for the conjugate  $U(n)$ -action, denoted by  $\mathcal{O}_\lambda$ , has a stabilizer isomorphic to  $U(k_1) \times \dots \times U(k_{r+1})$  and hence we get

$$\mathcal{O}_\lambda \cong U(n)/U(k_1) \times \dots \times U(k_{r+1}) \cong \mathcal{F}\ell(\mathbf{n}).$$

In particular, we have

$$\dim_{\mathbb{R}} \mathcal{F}\ell(\mathbf{n}) = n^2 - \sum_{i=1}^{r+1} k_i^2.$$

Together with the Kirillov-Kostant-Souriau symplectic form  $\omega_\lambda$  on the co-adjoint orbit  $\mathcal{O}_\lambda$ , we get a symplectic manifold  $(\mathcal{O}_\lambda, \omega_\lambda)$  diffeomorphic to  $\mathcal{F}\ell(\mathbf{n})$ . Then the GC-polytope  $\mathcal{P}_\lambda$  is equal to the image of the following map

$$\begin{aligned} \Phi_\lambda &: \mathcal{F}\ell(\mathbf{n}) \rightarrow \mathbb{R}^d \\ X &\mapsto (\lambda_i^{(j)}(X))_{i,j} \end{aligned}$$

where  $\{\lambda_i^{(j)}\}_{i+j=\ell \geq 2}$  are eigenvalues of  $(\ell-1) \times (\ell-1)$  principal minor  $X^{(\ell-1)}$  of  $X \in \mathcal{H}$  satisfying

$$\lambda_1^{(\ell-1)}(X) \geq \lambda_2^{(\ell-2)}(X) \geq \dots \geq \lambda_{\ell-1}^{(1)}(X)$$

for each  $\ell = 2, \dots, n$ . Guillemin and Sternberg [GS] proved that the map  $\Phi_\lambda$  is a completely integrable system on  $(\mathcal{O}_\lambda, \omega_\lambda)$ , called a *Gelfand-Cetlin system*, see [GS] for more details.

**1.2. Ladder diagrams.** In this paper, we study a combinatorial structure on GC-polytopes. More precisely, we study the *face lattice* of  $\mathcal{P}_\lambda$ , denoted by  $\mathcal{F}(\mathcal{P}_\lambda)$ , which consists of all faces of  $\mathcal{P}_\lambda$  graded by their geometric dimensions, and is equipped with the order relation given by the relation of inclusion of faces of  $\mathcal{P}_\lambda$ .

Our first aim is to describe the face lattice of a GC-polytope in terms of a *ladder diagram*. To define a ladder diagram, we first define  $Q^+$  to be the infinite directed graph with vertex set

$$V(Q^+) := \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0},$$

such that  $((i, j), (i', j'))$  is a directed edge if and only if  $(i', j') = (i, j + 1)$  or  $(i', j') = (i + 1, j)$ .

**Definition 1.1.** For a given positive integer  $n$ , let  $\mathbf{k} = (k_1, \dots, k_s)$  be a sequence of positive integers such that  $\sum_{i=1}^s k_i = n$ . Let  $n_i = \sum_{1 \leq j \leq i} k_j$  for  $i = 1, \dots, s$  with  $n_0 = 0$  and let

$$T_{\mathbf{k}} = \{(n_0, n - n_0), (n_1, n - n_1), \dots, (n_s, n - n_s)\} \subset V(Q^+).$$

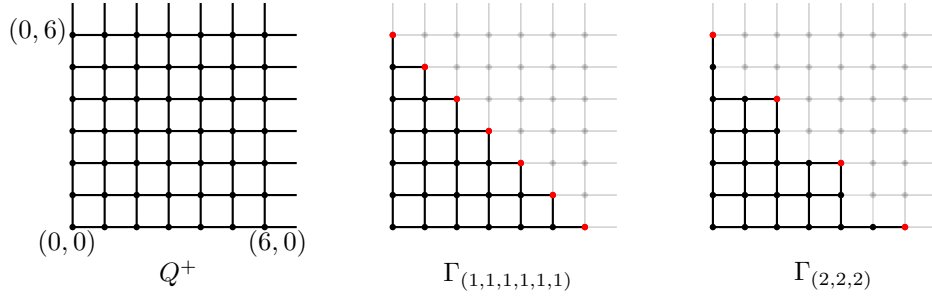
(1) The ladder diagram  $\Gamma_{\mathbf{k}}$  is defined as the induced subgraph of  $Q^+$  with vertex set

$$V(\Gamma_{\mathbf{k}}) = \{(a, b) \in V(Q^+) \mid a \leq c, b \leq d \text{ for some } (c, d) \in T_{\mathbf{k}}\}.$$

In other words, for two vertices  $(a, b)$  and  $(c, d)$  of  $\Gamma_{\mathbf{k}}$ ,  $((a, b), (c, d))$  is an edge of  $\Gamma_{\mathbf{k}}$  if and only if it is an edge of  $Q^+$ .

- (2)  $(0, 0) \in V(Q^+)$  is called the *origin* of  $\Gamma_{\mathbf{n}}$ .
- (3) A vertex  $v \in T_{\mathbf{k}}$  is called a *terminal vertex* of  $\Gamma_{\mathbf{k}}$ .
- (4) A vertex  $v \in V(\Gamma_{\mathbf{k}})$  is called *extremal* if  $v$  is either a terminal vertex or the origin, and *non-extremal* otherwise.

**Example 1.2.** The graphs  $Q^+$ ,  $\Gamma_{(1,1,1,1,1,1)}$ , and  $\Gamma_{(2,2,2)}$  are given as follows.



The red dots denote the terminal vertices for each graph.

**Remark 1.3.** Note that we defined a ladder diagram  $\Gamma_{\mathbf{k}}$  for a sequence  $\mathbf{k}$  of positive integers. However, this definition of  $\Gamma_{\mathbf{k}}$  can be naturally extended for all sequences of non-negative integers such that

$$\Gamma_{\mathbf{k}} := \Gamma_{\underline{\mathbf{k}}}$$

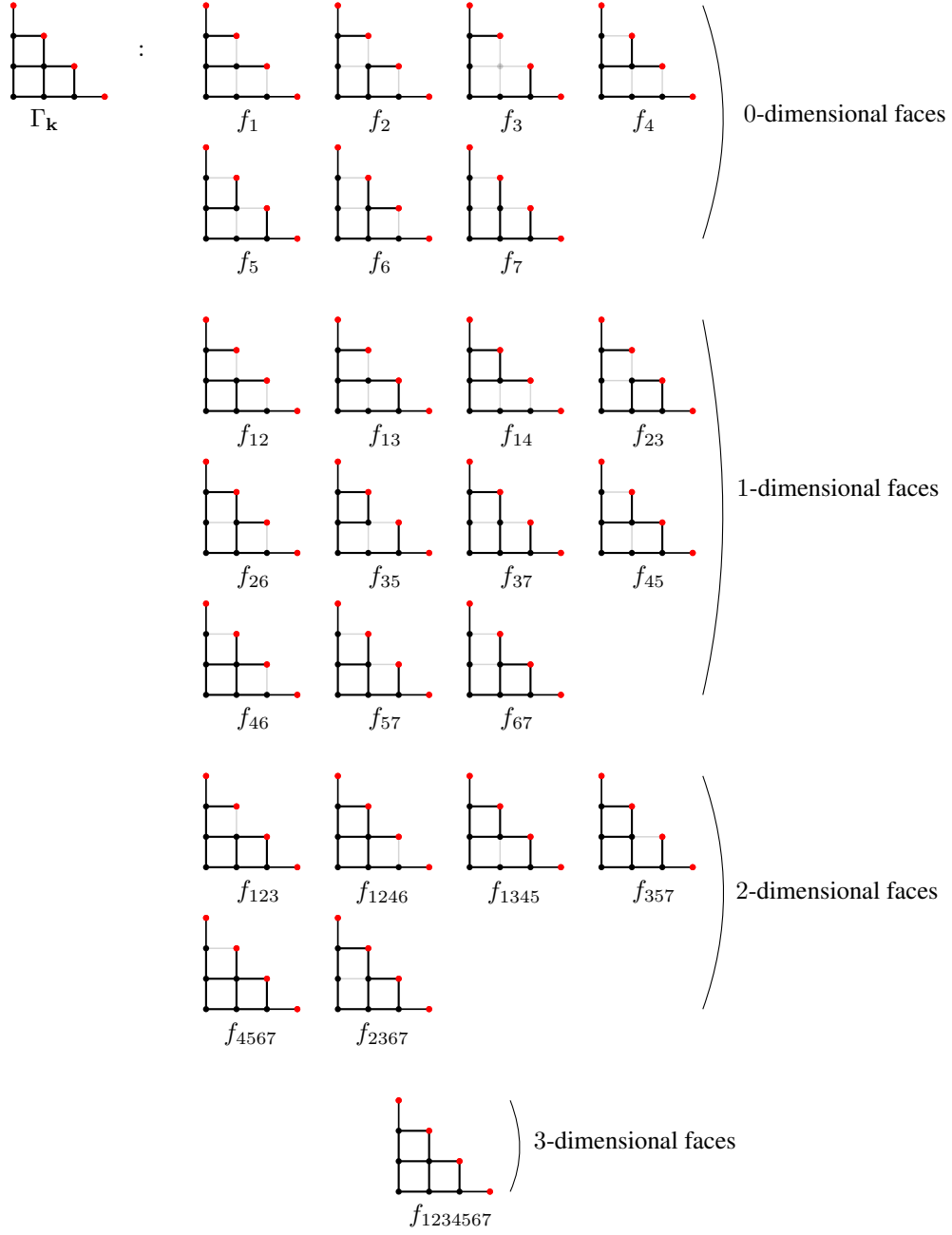
where  $\underline{\mathbf{k}}$  is the maximal subsequence of  $\mathbf{k}$  whose components are all positive.

**Definition 1.4** (Definition 2.2.2 in [BCKV]). A *positive path* on  $\Gamma_{\mathbf{k}}$  is a shortest path from the origin to a terminal vertex of  $\Gamma_{\mathbf{k}}$ .

**Definition 1.5** (A face structure on  $\Gamma_{\mathbf{k}}$ ). Let  $\gamma$  be a subgraph of  $\Gamma_{\mathbf{k}}$ .

- (1)  $\gamma$  is called a *face* of  $\Gamma_{\mathbf{k}}$  if
  - $V(\gamma)$  contains all terminal vertices of  $\Gamma_{\mathbf{k}}$ , and
  - $\gamma$  can be presented as a union of positive paths.
- (2) For two faces  $\gamma$  and  $\gamma'$  of  $\Gamma_{\mathbf{k}}$ , we say that  $\gamma$  is a *face* of  $\gamma'$  if  $\gamma \subset \gamma'$ .
- (3) A *dimension* of a face  $\gamma$  is defined by  $\dim \gamma := \text{rank } H_1(\gamma)$  by regarding  $\gamma$  as a one-dimensional CW-complex. In other words,  $\dim \gamma$  is the number of minimal cycles in  $\gamma$ .

We denote by  $\mathcal{F}(\Gamma_{\mathbf{k}})$  the set of all faces of  $\Gamma_{\mathbf{k}}$ . Then the face relation defined in (2) makes  $\mathcal{F}(\Gamma_{\mathbf{k}})$  a poset. In fact  $\mathcal{F}(\Gamma_{\mathbf{k}})$  is a lattice, see Remark 1.8. We call  $\mathcal{F}(\Gamma_{\mathbf{k}})$  the *face lattice* of  $\Gamma_{\mathbf{k}}$ .

FIGURE 1. The faces of  $\Gamma_{(1,1,1)}$ .

*Remark 1.6.* Let  $\gamma$  be a face of  $\Gamma_{\mathbf{k}}$  and let  $v$  be a non-extremal vertex in  $V(\gamma)$ . Figure 4 illustrates the impossible types of the set of edges in  $\gamma$  incident to  $v$ .

Note that  $\Gamma_{\mathbf{k}}$  itself is a face of  $\Gamma_{\mathbf{k}}$  of maximal dimension, and we have

$$\dim \Gamma_{\mathbf{k}} = \text{rank } H_1(\Gamma_{\mathbf{k}}) = \sum_{1 \leq i < j \leq s} k_i k_j = \frac{1}{2} \left( n^2 - \sum_{i=1}^s k_i^2 \right).$$

**Example 1.7.** Let  $\mathbf{k} = (1, 1, 1)$ . Then we can classify all faces of  $\Gamma_{\mathbf{k}}$  as in Figure 1. There are 7 faces of dimension zero, 11 faces of dimension one, 6 faces of dimension two, and 1 face of dimension three in  $\Gamma_{\mathbf{k}}$  as we see in Figure 1. For faces  $f_I$  and  $f_J$  with  $I, J \subset \{1, 2, \dots, 7\}$ , we can easily check that  $f_I$  is a face of  $f_J$  if and only if  $I \subset J$ . In particular, we have  $f_J = \cup_{j \in J} f_j$ .

**Remark 1.8.** By definition, a union of faces of  $\Gamma_{\mathbf{k}}$  is again a face of  $\Gamma_{\mathbf{k}}$ . In fact, if  $\gamma_1, \dots, \gamma_\ell$  are faces of  $\Gamma_{\mathbf{k}}$ , then  $\bigcup_{i=1}^\ell \gamma_i$  is the smallest face containing all  $\gamma_i$ 's. Thus the union  $\bigcup$  plays the role of the *join operator*  $\vee$  for a lattice. On the other hand, the intersection of faces need not be a face. For example,  $f_{123} \cap f_{357}$  in Figure 1 cannot be expressed as a union of positive paths, and hence it is not a face of  $\Gamma_{\mathbf{k}}$  by Definition 1.5. However, there is a unique maximal face  $f_3$  contained in  $f_{123} \cap f_{357}$ . Thus one can define the *meet*  $\gamma \wedge \gamma'$  of two faces of  $\Gamma_{\mathbf{k}}$  as the maximal face contained in the intersection  $\gamma \cap \gamma'$ . Then  $\mathbf{F}(\Gamma_{\mathbf{k}})$  becomes a lattice together with the join  $\vee$  and the meet  $\wedge$ .

The first part of our main theorem is as follows.

**Theorem 1.9.** Let  $\mathbf{k} = (k_1, \dots, k_s)$  be a sequence of positive integers and  $\mathbf{n} = (n_0, \dots, n_s)$  where  $n_i = \sum_{j=1}^i k_j$  for  $i = 1, \dots, s$  with  $n_0 = 0$ . Suppose that  $\lambda = (\lambda_1, \dots, \lambda_n)$  is a sequence of real numbers satisfying

$$\lambda_1 = \dots = \lambda_{n_1} > \dots > \lambda_{n_{s-1}+1} = \dots = \lambda_{n_s}$$

Then there is an isomorphism  $\phi$  between lattices

$$\phi : \mathcal{F}(\mathcal{P}_\lambda) \rightarrow \mathcal{F}(\Gamma_{\mathbf{k}})$$

such that  $\dim \phi(F) = \dim F$  for all  $F \in \mathcal{F}(\mathcal{P}_\lambda)$ .

Note that Theorem 1.9 is equivalent to say that there exists a bijective map

$$\{\text{faces of } \Gamma_{\mathbf{k}}\} \xrightarrow{\phi} \{\text{faces of } \mathcal{P}_\lambda\}$$

such that

- (1)  $\dim \phi(F) = \dim F$ , and
- (2)  $F \subset F' \Leftrightarrow \phi(F) \subset \phi(F')$

for every faces  $F$  and  $F'$  of  $\mathcal{P}_\lambda$ . In particular,  $\phi$  preserves the operators  $\vee$  and  $\wedge$ .

**Example 1.10.** Let  $\lambda = (2, 1, 0)$ . Then  $\mathcal{P}_\lambda$  is given as follows.

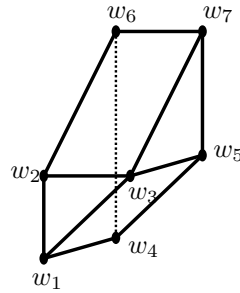


FIGURE 2. The GC-polytope  $\mathcal{P}_\lambda$  for  $\lambda = (2, 1, 0)$ .

We label each vertex of  $\mathcal{P}_\lambda$  with  $w_i$  for  $i \in \{1, \dots, 7\}$  as given in Figure 2. Similarly, we label each face of  $\mathcal{P}_\lambda$  with  $w_J$  for  $J \subset \{1, \dots, 7\}$  such that  $j \in J$  if and only if  $w_J$  contains  $w_j$ . Then we can easily check that

$$\begin{aligned} \phi : \mathcal{F}(\Gamma_{\mathbf{k}}) &\longrightarrow \mathcal{F}(\mathcal{P}_\lambda) \\ f_J &\longmapsto w_J \end{aligned}$$

is an isomorphism where  $f_J$  denotes a face of  $\Gamma_{\mathbf{k}}$  defined in 1.7.

**Remark 1.11.** Note that Theorem 1.9 tells us that the face lattice  $\mathcal{F}(\mathcal{P}_\lambda)$  of  $\mathcal{P}_\lambda$  depends only on  $\mathbf{k}$ .

**1.3. Exponential generating functions of  $f$ -polynomials.** The second aim of this article is to study  $f$ -vectors of GC-polytopes by using Theorem 1.9.

Let  $\mathbf{k}$  be a sequence of non-negative integers and let  $\Gamma_{\mathbf{k}}$  be the corresponding ladder diagram in the sense of Remark 1.3. Let  $f_i(\mathbf{k})$  be the number of faces of  $\Gamma_{\mathbf{k}}$  of dimension  $i$  for  $i = 0, 1, \dots, \dim \Gamma_{\mathbf{k}}$ . We call  $f(\mathbf{k}) := (f_0(\mathbf{k}), \dots, f_{\dim \Gamma_{\mathbf{k}}}(\mathbf{k}))$  the  $f$ -vector of  $\Gamma_{\mathbf{k}}$ . Then the  $f$ -polynomial  $\mathbf{F}_{\mathbf{k}}(t)$  of  $\Gamma_{\mathbf{k}}$  is defined by

$$\mathbf{F}_{\mathbf{k}}(t) := \sum_{i=0}^{\dim \Gamma_{\mathbf{k}}} f_i(\mathbf{k}) t^i = \sum_{\gamma \in \mathcal{F}(\Gamma_{\mathbf{k}})} t^{\dim \gamma},$$

where  $t$  is a formal parameter. In particular, the number of zero-dimensional faces of  $\Gamma_{\mathbf{k}}$ , denoted by  $\mathbf{V}_{\mathbf{k}}$ , is equal to  $\mathbf{F}_{\mathbf{k}}(0)$ .

For each positive integer  $s$ , we define the power series  $\Psi_s$  in formal variables  $x_1, \dots, x_s$ , and  $t$  as

$$(1.2) \quad \Psi_0(t) := 1, \quad \Psi_s(x_1, \dots, x_s; t) := \sum_{k_1, \dots, k_s \geq 0} \mathbf{F}_{(k_1, \dots, k_s)}(t) \frac{x_1^{k_1} \cdots x_s^{k_s}}{k_1! \cdots k_s!}.$$

For the sake of simplicity, we denote by

$$(1.3) \quad \Psi_s(\mathbf{x}; t) = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^s} \mathbf{F}_{\mathbf{k}}(t) \frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}$$

where  $\mathbf{x} = (x_1, \dots, x_s)$ ,  $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} \cdots x_s^{k_s}$ , and  $\mathbf{k}! = k_1! \cdots k_s!$ . Then we can prove the following.

**Theorem 1.12.** *The following equation*

$$(\mathcal{D}_s(\Psi_{2s-1}(\mathbf{x} * \mathbf{y}; t)))|_{\mathbf{y}=0} = 0$$

holds for every positive integer  $s$  where

$$\mathbf{x} * \mathbf{y} = (x_1, y_1, \dots, x_{s-1}, y_{s-1}, x_s)$$

for  $\mathbf{x} = (x_1, \dots, x_s)$  and  $\mathbf{y} = (y_1, \dots, y_{s-1})$ , and

$$\mathcal{D}_s = \frac{\partial^s}{\partial x_1 \cdots \partial x_s} - \prod_{i=1}^{s-1} \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_{i+1}} + t \cdot \frac{\partial}{\partial y_i} \right).$$

**1.4. Theorem of Gusev-Kiritchenko-Timorin.** As a corollary of Theorem 1.12, we obtain the following result proved by Gusev, Kiritchenko, and Timorin, see also [GKT, Theorem 1.1].

**Corollary 1.13.** [GKT] For  $s \geq 1$ , let

$$E_s(\mathbf{x}) := \Psi_s(\mathbf{x}; 0) = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^s} \mathbf{V}_{\mathbf{k}} \frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}.$$

where  $\mathbf{V}_{\mathbf{k}} := \mathbf{F}_{\mathbf{k}}(0)$  is the number of vertices of  $\Gamma_{\mathbf{k}}$ . Then  $E_s(\mathbf{x})$  is a solution of the following partial differential equation

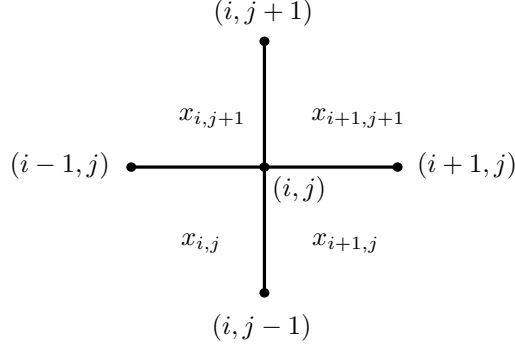
$$\left( \frac{\partial^s}{\partial x_1 \cdots \partial x_s} - \prod_{i=1}^{s-1} \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_{i+1}} \right) \right) E_s(\mathbf{x}) = 0.$$

*Proof.* For  $s \geq 1$ , let us denote by

$$\mathcal{D}'_s = \frac{\partial^s}{\partial x_1 \cdots \partial x_s} - \prod_{i=1}^{s-1} \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_{i+1}} \right).$$

Then  $\mathcal{D}_s = \mathcal{D}'_s + t \cdot \mathcal{D}''_s$  for some partial differential operator  $\mathcal{D}''_s$ . Observe that

- $\Psi_s(\mathbf{x}; 0) = E_s(\mathbf{x})$ ,
- $\Psi_s(\mathbf{x}; t) = \Psi_{2s-1}(\mathbf{x} * \mathbf{0}; t) = \Psi_{2s-1}(\mathbf{x} * \mathbf{y}; t)|_{\mathbf{y}=0}$ , and
- $\mathcal{D}'_s(\Psi_s(\mathbf{x}; t))|_{t=0} = \mathcal{D}'_s(\Psi_s(\mathbf{x}; 0))$ .

FIGURE 3. Coordinates  $x_I$  of  $\mathbb{R}^d$ 

Then by Theorem 1.12, we obtain

$$\begin{aligned} 0 &= (\mathcal{D}_s(\Psi_{2s-1}(\mathbf{x} * \mathbf{y}; t)))|_{\mathbf{y}=0} \\ &= (\mathcal{D}'_s(\Psi_{2s-1}(\mathbf{x} * \mathbf{y}; t)))|_{\mathbf{y}=0} + (t \cdot \mathcal{D}''_s(\Psi_{2s-1}(\mathbf{x} * \mathbf{y}; t)))|_{\mathbf{y}=0} \\ &= \mathcal{D}'_s(\Psi_s(\mathbf{x}; t)) + (t \cdot \mathcal{D}''_s(\Psi_{2s-1}(\mathbf{x} * \mathbf{y}; t)))|_{\mathbf{y}=0} \end{aligned}$$

for every  $t \in \mathbb{R}$ . Thus by substituting  $t = 0$ , we have

$$\mathcal{D}'_s(\Psi_s(\mathbf{x}; t))|_{t=0} = \mathcal{D}'_s(\Psi_s(\mathbf{x}; 0)) = \mathcal{D}'_s(E_s(\mathbf{x})) = 0.$$

which completes the proof.  $\square$

*Remark 1.14.* Finding a partial differential equation whose solution is the exponential generating function of  $f$ -polynomials of GC-polytopes was an open problem posed by Gusev, Kiritchenko, and Timorin in [GKT]. Thus Theorem 1.12 gives the answer for the problem.

This paper is organized as follows. In Section 2, we give the proof of Theorem 1.9. And in Section 3, we give the proof of Theorem 1.12.

## 2. FACE LATTICES OF LADDER DIAGRAMS

In this section, we study face lattices of ladder diagrams defined in Section 1 and prove Theorem 1.9.

Let us fix an integer  $n > 1$ , a sequence  $\lambda = (\lambda_1, \dots, \lambda_n)$  of real numbers satisfying

$$\lambda_1 = \dots = \lambda_{n_1} > \lambda_{n_1+1} = \dots = \lambda_{n_2} > \dots > \lambda_{n_r+1} = \dots = \lambda_n,$$

and a sequence  $\mathbf{k} = (k_1, \dots, k_{r+1})$  with  $k_i = n_i - n_{i-1}$  for  $i = 1, 2, \dots, r+1$ , where  $n_0 = 0$  and  $n_{r+1} = n$ .

Let  $I = \{(i, j) \in \mathbb{Z}^2 \mid i, j \geq 1, i + j \leq n\}$  be an index set with  $|I| = d := \binom{n}{2}$ . As in (1.1), we denote the coordinates of  $\mathbb{R}^d$  by  $x_I = (x_{i,j})_{(i,j) \in I} \in \mathbb{R}^d$  so that  $\mathcal{P}_\lambda$  is written by

$$\mathcal{P}_\lambda = \{x_I \mid x_{i,j+1} \geq x_{i,j} \geq x_{i+1,j} \text{ for } (i, j) \in I\},$$

where  $x_{i,n+1-i} = \lambda_i$  for  $i = 1, 2, \dots, n$ .

For each face  $F \in \mathcal{F}(\mathcal{P}_\lambda)$ , let us define the subgraph  $\phi(F)$  of  $Q^+$  whose edge set is

$$\begin{aligned} E(\phi(F)) &= \{((0, i), (0, i+1)) \mid 0 \leq i \leq n-1\} \cup \{((i, 0), (i+1, 0)) \mid 0 \leq i \leq n-1\} \\ &\quad \cup \{((i-1, j), (i, j)) \mid \text{if there is a point } x_I \in F \text{ with } x_{i,j} < x_{i,j+1}\} \\ &\quad \cup \{((i, j-1), (i, j)) \mid \text{if there is a point } x_I \in F \text{ with } x_{i,j} > x_{i+1,j}\}, \end{aligned}$$

and vertex set  $V(\phi(F))$  is defined to be a subset of  $V(Q^+)$  whose element is an endpoint of an edge in  $E(\phi(F))$ . See Figure 3 for an illustration of the possible sets of edges incident to the vertex  $(i, j) \in V(\Gamma_{\mathbf{k}})$  and the coordinates  $x_{i,j}$ 's for each  $(i, j) \in I$  (cf. (1.1)).

**Lemma 2.1.**  $\phi(F)$  is a subgraph of  $\Gamma_{\mathbf{k}}$ .

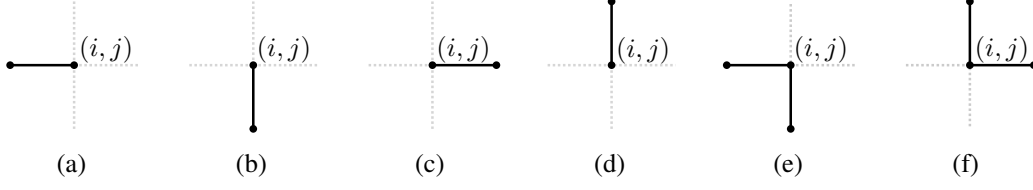


FIGURE 4. The impossible sets of edges in a face of  $\Gamma_{\mathbf{k}}$  incident to non-extremal vertex  $(i, j) \in V(\Gamma_{\mathbf{k}})$ .

*Proof.* It is enough to show that each edge of  $E(\phi(F))$  is lying on  $\Gamma_{\mathbf{k}}$ . Let  $e = ((i-1, j), (i, j))$  be any horizontal edge in  $Q^+$  not lying on  $\Gamma_{\mathbf{k}}$ . Then by Definition 1.1, there is no terminal vertex  $(c, d) \in T_{\mathbf{k}}$  such that  $i \leq c$  and  $j \leq d$ . Equivalently,  $(i, j) \notin V(\Gamma_{\mathbf{k}})$  so that there exist consecutive terminal vertices  $(n_{\ell}, n - n_{\ell})$  and  $(n_{\ell+1}, n - n_{\ell+1})$  for some  $0 \leq \ell \leq r$  such that

$$n_{\ell} < i < n_{\ell+1}, \quad \text{and} \quad n - n_{\ell+1} < j < n - n_{\ell}.$$

Then we have  $x_{i+1,j} = x_{i,j} = x_{i,j+1} = \lambda_{n_{\ell+1}}$  by (1.1) and hence  $e$  cannot be lying on  $\phi(F)$  by definition of  $\phi$ , i.e., any edge of  $\phi(F)$  is lying on  $\Gamma_{\mathbf{k}}$  for any  $F \in \mathcal{F}(\mathcal{P}_{\lambda})$ . Similarly, we can easily see that the same argument holds for a vertical edge of  $Q^+$  so that  $\phi(F)$  is a subgraph of  $\Gamma_{\mathbf{k}}$  for every  $F \in \mathcal{F}(\mathcal{P}_{\lambda})$ .  $\square$

**Lemma 2.2.**  $\phi(F)$  contains every terminal vertex and the origin of  $\Gamma_{\mathbf{k}}$ .

*Proof.* It is clear that  $\phi(F)$  contains the origin and two terminal vertices  $(0, n)$  and  $(n, 0)$  by definition of  $E(\phi(F))$  and  $V(\phi(F))$ . Now, let us suppose that a terminal vertex  $(n_{\ell}, n - n_{\ell})$  of  $\Gamma_{\mathbf{k}}$  is not contained in  $\phi(F)$  for some  $1 \leq \ell \leq r$ . Then we can see that two edges  $((n_{\ell} - 1, n - n_{\ell}), (n_{\ell}, n - n_{\ell}))$  and  $((n_{\ell}, n - n_{\ell} - 1), (n_{\ell}, n - n_{\ell}))$  are not in  $E(\phi(F))$  which implies that any point  $x_I \in F$  satisfies  $\lambda_{N_{\ell}} = x_{n_{\ell}, n - n_{\ell}} = \lambda_{n_{\ell+1}}$ , and this contradicts  $\lambda_{n_{\ell}} > \lambda_{n_{\ell+1}}$ . Therefore,  $\phi(F)$  contains every terminal vertices of  $\Gamma_{\mathbf{k}}$ .  $\square$

**Lemma 2.3.**  $\phi(F)$  does not contain a non-extremal<sup>1</sup> vertex  $(i, j)$  which is one of six types in Figure 4. Equivalently, every edge  $e \in E(\phi(F))$  can be extended to a positive path lying on  $\phi(F)$ .

*Proof.* It is straightforward by (1.1).  $\square$

By Lemma 2.1, Lemma 2.2, and Lemma 2.3, we have the following corollary.

**Corollary 2.4.** For any  $F \in \mathcal{F}(\mathcal{P}_{\lambda})$ , we have  $\phi(F) \in \mathcal{F}(\Gamma_{\mathbf{k}})$ .

**Lemma 2.5.** The map  $\phi : \mathcal{F}(\mathcal{P}_{\lambda}) \rightarrow \mathcal{F}(\Gamma_{\mathbf{k}})$  is a bijection.

*Proof.* We will show this by constructing an inverse of  $\phi$ . Let  $\gamma \in \mathcal{F}(\Gamma_{\mathbf{k}})$ . Then we define  $\psi(\gamma)$  to be the set of points  $x_I \in \mathcal{P}_{\lambda}$  such that

- if  $((i-1, j), (i, j)) \notin \gamma$ , then  $x_{i,j} = x_{i,j+1}$ , and
- if  $((i, j-1), (i, j)) \notin \gamma$ , then  $x_{i,j} = x_{i+1,j}$ .

It is obvious that  $\psi(\gamma) \in \mathcal{F}(\mathcal{P}_{\lambda})$  since  $\psi(\gamma)$  is the intersection of  $\mathcal{P}_{\lambda}$  and some facet hyperplanes determined by the above equalities.

Let  $\gamma' = \phi(\psi(\gamma))$ . In order to show that  $\psi$  is the inverse of  $\phi$ , we need to show  $\gamma' = \gamma$ . In fact, we only need to show  $E(\gamma') = E(\gamma)$ . From the construction of  $\psi(\gamma)$ , every edge not in  $\gamma$  is not in  $\gamma'$ , which implies  $E(\gamma') \subseteq E(\gamma)$ . Thus it remains to show that  $E(\gamma) \subseteq E(\gamma')$ .

Let us consider the point  $\bar{x}_I = (\bar{x}_{i,j})_{(i,j) \in I} \in \psi(\gamma)$  defined recursively as follows:

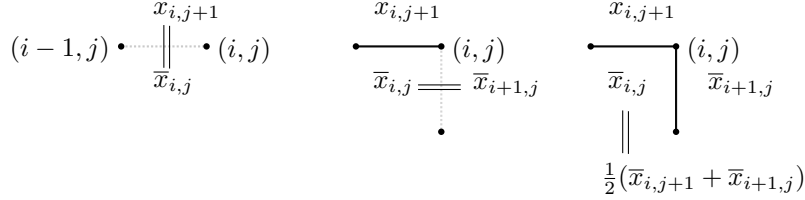
- Set  $\bar{x}_{i,n+1-i} = \lambda_i$  for  $i = 1, 2, \dots, n$ .

<sup>1</sup>See Definition 1.1 (4).



- If  $\bar{x}_{i,j+1}, \bar{x}_{i+1,j}$  are defined, then  $\bar{x}_{i,j}$  is defined by

$$\bar{x}_{i,j} = \begin{cases} \bar{x}_{i,j+1} & \text{if } ((i-1, j), (i, j)) \notin E(\gamma), \\ \bar{x}_{i+1,j} & \text{if } ((i-1, j), (i, j)) \in E(\gamma) \text{ and } ((i, j-1), (i, j)) \notin E(\gamma), \\ \frac{1}{2}(\bar{x}_{i,j+1} + \bar{x}_{i+1,j}) & \text{if } ((i-1, j), (i, j)) \in E(\gamma) \text{ and } ((i, j-1), (i, j)) \in E(\gamma). \end{cases}$$



Then we claim that

**C1:** If  $((i-1, i), (i, j)) \in E(\gamma)$ , then  $\bar{x}_{i,j} < \bar{x}_{i,j+1}$ , and

**C2:** if  $((i, j-1), (i, j)) \in E(\gamma)$ , then  $\bar{x}_{i,j} > \bar{x}_{i+1,j}$ ,

which implies that  $E(\gamma) \subseteq E(\gamma')$ , which will finish the proof.

For the proof of the claim, suppose that it is false. Then we can find a lexicographically maximal<sup>2</sup> vertex  $(i, j)$  for which **C1** or **C2** is false. Suppose that **C1** is false. Then we have  $((i-1, j), (i, j)) \in E(\gamma)$  and  $\bar{x}_{i,j} = \bar{x}_{i,j+1}$ .

**CASE 1:**  $((i, j-1), (i, j)) \notin \gamma$ . By definition of  $\bar{x}_I$  and by our assumption, we have  $\bar{x}_{i,j} = \bar{x}_{i+1,j} = \bar{x}_{i,j+1}$ . If  $i+j = n$ , then we have  $\lambda_i = \bar{x}_{i,j+1} = \bar{x}_{i,j} = \bar{x}_{i+1,j} = \lambda_{i+1}$ . Then  $\lambda_i = \lambda_{i+1}$  implies that  $((i-1, j), (i, j)) \notin E(\Gamma_k)$  by definition of  $\Gamma_k$ , and hence  $((i-1, j), (i, j)) \notin E(\gamma)$  which contradicts the assumption that  $((i-1, j), (i, j)) \in E(\gamma)$ . Thus we must have  $i+j < n$ . Since  $\bar{x}_{i,j+1} \geq \bar{x}_{i+1,j+1} \geq \bar{x}_{i+1,j}$  and  $\bar{x}_{i,j+1} = \bar{x}_{i+1,j}$ , we have  $\bar{x}_{i,j+1} = \bar{x}_{i+1,j+1} = \bar{x}_{i+1,j}$ . Since we have taken  $(i, j)$  to be a maximal vertex (with respect to the lexicographic order) on which **C1** or **C2** fails, we have  $((i, j), (i, j+1)) \notin E(\gamma)$  and  $((i, j), (i+1, j)) \notin E(\gamma)$ . Then  $((i-1, j), (i, j))$  is the only edge incident to  $(i, j)$  in  $\gamma$ , which contradicts Lemma 2.3.

**CASE 2:**  $((i, j-1), (i, j)) \in \gamma$ . In this case, we have  $\bar{x}_{i,j} = \frac{1}{2}(\bar{x}_{i,j+1} + \bar{x}_{i+1,j})$ . Since  $\bar{x}_{i,j+1} \geq \bar{x}_{i,j} \geq \bar{x}_{i+1,j}$  and  $\bar{x}_{i,j} = \frac{1}{2}(\bar{x}_{i,j+1} + \bar{x}_{i+1,j}) = \bar{x}_{i,j+1}$  by our assumption, we have  $\bar{x}_{i,j+1} = \bar{x}_{i,j} = \bar{x}_{i+1,j}$ . Thus we may apply the same argument as in **CASE 1**, and hence we can deduce a contradiction.

Thus  $(i, j)$  satisfies **C1**. Similarly we can show that  $(i, j)$  also satisfies **C2**. Thus it completes the proof of our claim **C1** and **C2**.  $\square$

The following lemma completes the proof of Theorem 1.9.

**Lemma 2.6.** *The map  $\phi : \mathcal{F}(\mathcal{P}_\lambda) \rightarrow \mathcal{F}(\Gamma_k)$  is a poset isomorphism. Moreover, we have*

$$\dim F = \dim \phi(F)$$

for every  $F \in \mathcal{F}(\mathcal{P}_\lambda)$ .

*Proof.* Let  $F_1$  and  $F_2$  be faces of  $\mathcal{P}_\lambda$  such that  $F_1 \subseteq F_2$ . Then  $F_1$  is an intersection of  $F_2$  and some facet hyperplanes, i.e.,  $F_1$  is obtained from  $F_2$  by replacing some inequalities  $x_{i,j+1} \geq x_{i,j}$  or  $x_{i+1,j} \geq x_{i,j}$  by equalities  $x_{i,j+1} = x_{i,j}$  or  $x_{i+1,j} = x_{i,j}$ . By the definition of  $\phi$ , in this case  $\phi(F_1)$  is obtained from  $\phi(F_2)$  by removing corresponding edges. Thus we have  $\phi(F_1) \subseteq \phi(F_2)$ . Conversely, suppose that  $\phi(F_1) \subseteq \phi(F_2)$ . By the construction of the inverse map of  $\phi$  in the proof of Lemma 2.5, we clearly have  $F_1 \subseteq F_2$ . Thus  $\phi$  is a poset isomorphism.

For the dimension formula, recall that

$$\dim \mathcal{P}_\lambda = \dim \Gamma_k = \frac{1}{2} \left( n^2 - \sum_{i=1}^s k_i^2 \right).$$

<sup>2</sup>The lexicographic order on  $V(\Gamma_k)$  is defined by  $(i, j) \leq (i', j')$  if and only if  $i \leq i'$ , or  $i = i'$  and  $j \leq j'$ .

Since  $\phi$  is a poset isomorphism,  $\phi$  maps a maximal chain  $F_0 \subset F_1 \subset \cdots \subset F_{\dim P_\lambda}$  in  $\mathcal{F}(P_\lambda)$  to the maximal chain  $\phi(F_0) \subset \phi(F_1) \subset \cdots \subset \phi(F_{\dim P_\lambda})$  in  $\mathcal{F}(\Gamma_{\mathbf{k}})$ . Then the dimension formula follows from the simple observation that  $\gamma \subsetneq \gamma'$  implies  $\dim \gamma < \dim \gamma'$  for every  $\gamma, \gamma' \in \mathcal{F}(\Gamma_{\mathbf{k}})$ . □

### 3. EXPONENTIAL GENERATING FUNCTIONS AND PDE SYSTEMS

In this section we study the exponential generating function of  $f$ -polynomials of  $\Gamma_{\mathbf{k}}$ 's defined by

$$\Psi_s(\mathbf{x}; t) = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^s} \mathbf{F}_{\mathbf{k}}(t) \frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}$$

where  $\mathbf{F}_{\mathbf{k}}(t)$  is the  $f$ -polynomial of  $\Gamma_{\mathbf{k}}$ . Also, we give the complete proof of Theorem 1.12.

**3.1. Notations.** To begin with, we first introduce some notations as follows.

*Notation 3.1.* Let  $s \geq 1$  be an integer.

- For a multivariable  $\mathbf{x} = (x_1, \dots, x_s)$  and  $\mathbf{a} = (a_1, \dots, a_s) \in \mathbb{Z}_{\geq 0}^s$ ,

$$\mathbf{x}^{\mathbf{a}} := x_1^{a_1} \cdots x_s^{a_s}, \quad \mathbf{a}! := a_1! \cdots a_s!.$$

- For another multivariable  $\mathbf{y} = (y_1, \dots, y_{s-1})$ ,

$$\mathbf{x} * \mathbf{y} := (x_1, y_1, \dots, x_{s-1}, y_{s-1}, x_s).$$

In particular, we have

$$(\mathbf{x} * \mathbf{y})^{\mathbf{a} * \mathbf{b}} = \mathbf{x}^{\mathbf{a}} \mathbf{y}^{\mathbf{b}}, \quad (\mathbf{a} * \mathbf{b})! = \mathbf{a}! \cdot \mathbf{b}!.$$

*Notation 3.2.* Let  $W_{s-1}$  be the set of all sequences of length  $(s-1)$  on the set  $\{1, 0\}$ ,  $(0, 1)$ ,  $(1, 1)$ , i.e., each element of  $W_{s-1}$  is of the form

$$\mathbf{w} = ((\alpha_1, \beta_1), \dots, (\alpha_{s-1}, \beta_{s-1})), \quad (\alpha_i, \beta_i) \in \{(1, 0), (0, 1), (1, 1)\} \text{ for } 1 \leq i \leq s-1.$$

In particular, we have  $\#(W_{s-1}) = 3^{s-1}$ . For  $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{Z}^s$  and  $\mathbf{w} = ((\alpha_1, \beta_1), \dots, (\alpha_{s-1}, \beta_{s-1})) \in W_{s-1}$ , we denote by

- $\alpha_s = \beta_0 = 1$ ,
- $d_{\mathbf{w}}(\mathbf{k}) = (k'_1, \dots, k'_s) \in \mathbb{Z}^s$  where

$$k'_i = k_i - (1 - \alpha_i) - (1 - \beta_{i-1}), \quad 1 \leq i \leq s,$$

- $r_{\mathbf{w}}(\mathbf{k}) = (k''_1, \dots, k''_s) \in \mathbb{Z}^s$  where

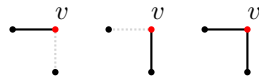
$$k''_i = k_i + 1 - \alpha_i - \beta_{i-1}, \quad 1 \leq i \leq s,$$

- $\tilde{\mathbf{w}} = (\alpha_1 \beta_1, \dots, \alpha_{s-1} \beta_{s-1})$ , and
- $|\mathbf{w}| = \sum_{i=1}^{s-1} \alpha_i \beta_i$ .

To help the readers understand the meaning of  $W_{s-1}$ ,  $d_{\mathbf{w}}(\mathbf{k})$ , and  $r_{\mathbf{w}}(\mathbf{k})$ , we briefly give an additional explanation as follows. Let  $\mathbf{k} = (k_1, \dots, k_s) \in (\mathbb{Z}_{\geq 1})^s$  so that the set of terminal vertices of  $\Gamma_{\mathbf{k}}$  is given by

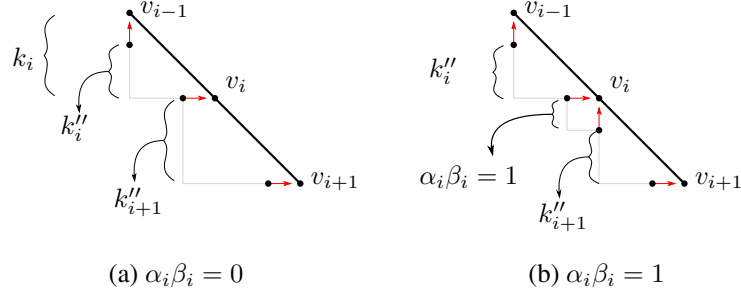
$$T_{\mathbf{k}} = \{v_i = (n_i, n - n_i) \in V(\Gamma_{\mathbf{k}}) \mid n_0 = 0, n_i = \sum_{j=1}^i k_j, i = 1, \dots, s\}.$$

For a face  $\gamma \in \Gamma_{\mathbf{k}}$ , the shape of  $\gamma$  near a vertex  $v_i \in T_{\mathbf{k}}$  for  $i \neq 0, s$  is one of three types :



Near  $v_0$  and  $v_s$ , the shape of  $\gamma$  is equal to



FIGURE 5. Definition of  $r_{\mathbf{w}}(\mathbf{k})$  and  $\tilde{\mathbf{w}}$ .

Thus the shape of  $\gamma$  near  $T_{\mathbf{k}}$  is determined by the following map

$$\mathcal{A}_{\gamma} : T_{\mathbf{k}} \rightarrow \{\rightarrow, \uparrow, \rightarrow\uparrow\}, \quad \mathcal{A}_{\gamma}(v_0) = \uparrow, \quad \mathcal{A}_{\gamma}(v_s) = \rightarrow,$$

called an *assignment on  $T_{\mathbf{k}}$* , which is defined in the obvious way, see Figure 6. Then we may identify  $W_{s-1}$  with the set of all assignments on  $T_{\mathbf{k}}$ , where  $\rightarrow$  corresponds to  $(1, 0)$ ,  $\uparrow$  corresponds to  $(0, 1)$ , and  $\rightarrow\uparrow$  corresponds to  $(1, 1)$ . Then  $\tilde{\mathbf{w}}$  is the vector which assigns the position of  $\rightarrow\uparrow$ 's, and  $|\mathbf{w}|$  is the number of  $\rightarrow\uparrow$ 's in  $\mathbf{w}$  for each  $\mathbf{w} \in W_{s-1}$ .

Now, let us think of the geometric meaning of  $r_{\mathbf{w}}(\mathbf{k})$  and  $d_{\mathbf{w}}(\mathbf{k})$ . For each  $\mathbf{w} = ((\alpha_1, \beta_1), \dots, (\alpha_{s-1}, \beta_{s-1})) \in W_{s-1}$ , let us consider a subgraph  $g_{\mathbf{w}}$  of  $\Gamma_{\mathbf{k}}$  such that the edge set of  $g_{\mathbf{w}}$  is defined to be

$$E(g_{\mathbf{w}}) := \{(v_i - (1, 0), v_i) \mid \alpha_i = 1\} \cup \{(v_i - (0, 1), v_i) \mid \beta_i = 1\},$$

and the vertex set  $V(g_{\mathbf{w}})$  is defined to be the set of endpoints of edges in  $E(g_{\mathbf{w}})$ .

It is easy to check that  $V(g_{\mathbf{w}}) = V_{n-1}(g_{\mathbf{w}}) \cup T_{\mathbf{k}}$ , where  $V_{n-1}(g_{\mathbf{w}})$  is the set of vertices of  $g_{\mathbf{w}}$  lying on the line whose equation is given by  $x + y = n - 1$  where  $n = \sum_{i=1}^s k_i$ . See Figure 6 for example : for each assignment  $\mathbf{w}$  on  $T_{\mathbf{k}}$ , the blue dots are the vertices in  $V_{n-1}(g_{\mathbf{w}})$ , the red dots are the vertices in  $T_{\mathbf{k}}$ , and the black line segments are edges of  $g_{\mathbf{w}}$ . Thus  $V(g_{\mathbf{w}})$  defines a unique ladder diagram, denoted by  $\Gamma_{\mathbf{k}}(\mathbf{w})$ , whose set of terminal vertices is equal to  $V_{n-1}(g_{\mathbf{w}})$ . Then the following lemma interprets the geometric meaning of  $r_{\mathbf{w}}(\mathbf{k})$ .

**Lemma 3.3.**  $\Gamma_{r_{\mathbf{w}}(\mathbf{k}) * \tilde{\mathbf{w}}} = \Gamma_{\mathbf{k}}(\mathbf{w})$ .

*Proof.* Note that each sequence  $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{Z}_{\geq 0}^s$  defines the unique ladder diagram  $\Gamma_{\mathbf{k}}$ . In particular, each  $k_i$  is the same as the difference of  $y$ -coordinates between two consecutive vertices  $v_{i-1}$  and  $v_i$  in  $T_{\mathbf{k}}$ .

As seen in Figure 5, each component  $k''_i$  of  $r_{\mathbf{w}}(\mathbf{k})$  measures the difference of the  $y$ -coordinates of two consecutive vertices in  $V_{n-1}(g_{\mathbf{w}})$  which corresponds to  $v_{i-1}$  and  $v_i$  in  $T_{\mathbf{k}}$ . In fact, we can easily see that  $k''_i$  is determined by the values  $\alpha_i$  and  $\beta_{i-1}$  and is equal to

$$k''_i = k_i + 1 - \alpha_i - \beta_{i-1}$$

in any case, see Figure 5.(a). Also, observe that each  $v_i$  with  $\alpha_i \beta_i = 1$  produces two vertices in  $V_{n-1}(g_{\mathbf{w}})$  such that the difference of their  $y$ -coordinates is 1, see Figure 5.(b). Thus the proof is straightforward.  $\square$

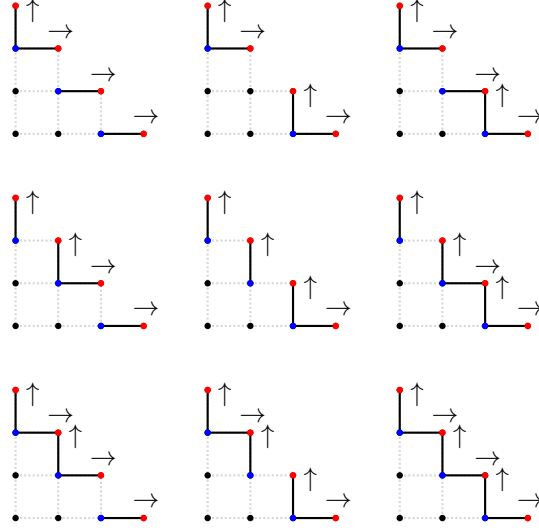
Finally, the meaning of  $d_{\mathbf{w}}(\mathbf{k})$  is given as follows.

**Lemma 3.4.** For any  $\mathbf{k} \in \mathbb{Z}^s$ , we have

$$r_{\mathbf{w}}(d_{\mathbf{w}}(\mathbf{k}) + \mathbf{1}) = \mathbf{k}$$

where  $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}_{\geq 1}^s$ . In particular, if  $d_{\mathbf{w}}(\mathbf{k}) \in \mathbb{Z}_{\geq 0}^s$ , then so is  $\mathbf{k}$ .

*Proof.* The proof is straightforward from the definitions of  $r_{\mathbf{w}}(\mathbf{k})$  and  $d_{\mathbf{w}}(\mathbf{k})$ .  $\square$

FIGURE 6.  $3^{3-1}$  possible types of edges near  $T_{(1,1,1)}$ .

**3.2. Partial differential operators.** For each  $\mathbf{w} = ((\alpha_1, \beta_1), \dots, (\alpha_{s-1}, \beta_{s-1})) \in W_{s-1}$ , let us denote by

$$D_{\mathbf{w}} = \prod_{i=1}^{s-1} \left( \frac{\partial}{\partial x_i} \right)^{1-\alpha_i} \left( \frac{\partial}{\partial x_{i+1}} \right)^{1-\beta_i} \left( t \cdot \frac{\partial}{\partial y_i} \right)^{\alpha_i \beta_i}$$

the differential operator defined on the ring of formal power series  $\mathbb{Q}[[x_1, y_1, \dots, x_{s-1}, y_{s-1}, x_s, t]]$ .

**Lemma 3.5.** *The following identity holds :*

$$\prod_{i=1}^{s-1} \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_{i+1}} + t \cdot \frac{\partial}{\partial y_i} \right) = \sum_{\mathbf{w} \in W_{s-1}} D_{\mathbf{w}}.$$

*Proof.* Let  $\mathbf{w} = ((\alpha_1, \beta_1), \dots, (\alpha_{s-1}, \beta_{s-1})) \in W_{s-1}$ . Then in each  $i$ th factor of the left hand side, we take  $\frac{\partial}{\partial x_i}$  if  $(\alpha_i, \beta_i) = (0, 1)$ ,  $\frac{\partial}{\partial x_{i+1}}$  if  $(\alpha_i, \beta_i) = (1, 0)$ , and  $t \cdot \frac{\partial}{\partial y_i}$  if  $(\alpha_i, \beta_i) = (1, 1)$ . Multiplying the chosen factors give  $D_{\mathbf{w}}$ . Thus the expansion of the left hand side is equal to the right hand side.  $\square$

**Lemma 3.6.** *Let  $\mathbf{k} \in \mathbb{Z}_{\geq 0}^s$ ,  $\mathbf{e} \in \mathbb{Z}_{\geq 0}^{s-1}$  and  $\mathbf{w} \in W_{s-1}$ . Then*

$$\left( D_{\mathbf{w}} \left( \frac{(\mathbf{x} * \mathbf{y})^{\mathbf{k} * \mathbf{e}}}{(\mathbf{k} * \mathbf{e})!} \right) \right) \Big|_{\mathbf{y}=0} = \begin{cases} t^{|\mathbf{w}|} \cdot \frac{\mathbf{x}^{d_{\mathbf{w}}(\mathbf{k})}}{d_{\mathbf{w}}(\mathbf{k})!}, & \text{if } \mathbf{e} = \tilde{\mathbf{w}} \text{ and } d_{\mathbf{w}}(\mathbf{k}) \in \mathbb{Z}_{\geq 0}^s, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* Note that the order of  $\frac{\partial}{\partial x_i}$  in  $D_{\mathbf{w}}$  is  $2 - \alpha_i - \beta_{i-1}$ , and the order of  $\frac{\partial}{\partial y_i}$  is  $\alpha_i \beta_i$  for each  $i = 1, \dots, s$ . Thus

$$\begin{aligned} D_{\mathbf{w}} \left( \frac{(\mathbf{x} * \mathbf{y})^{\mathbf{k} * \mathbf{e}}}{(\mathbf{k} * \mathbf{e})!} \right) \Big|_{\mathbf{y}=0} = 0 & \Leftrightarrow e_i \neq \alpha_i \beta_i \quad \text{or} \quad k_i < 2 - \alpha_i - \beta_{i-1} \text{ for some } 1 \leq i \leq s \\ & \Leftrightarrow \mathbf{e} \neq \tilde{\mathbf{w}} \quad \text{or} \quad d_{\mathbf{w}}(\mathbf{k}) \notin \mathbb{Z}_{\geq 0}^s. \end{aligned}$$

For the other case, we have

$$\begin{aligned} D_{\mathbf{w}} \left( \frac{(\mathbf{x} * \mathbf{y})^{\mathbf{k} * \mathbf{e}}}{(\mathbf{k} * \mathbf{e})!} \right) &= D_{\mathbf{w}} \left( \prod_{i=1}^s \frac{x_i^{k_i}}{k_i!} \prod_{j=1}^{s-1} \frac{y_j^{e_j}}{e_j!} \right) \\ &= \prod_{i=1}^s \frac{x_i^{k_i - 2 + \alpha_i + \beta_{i-1}}}{(k_i - 2 + \alpha_i + \beta_{i-1})!} \prod_{j=1}^{s-1} \frac{y_j^{e_j - \alpha_j \beta_j}}{(e_j - \alpha_j \beta_j)!} \cdot t^{\alpha_j \beta_j} \\ &= t^{|\mathbf{w}|} \cdot \frac{\mathbf{x}^{d_{\mathbf{w}}(\mathbf{k})}}{d_{\mathbf{w}}(\mathbf{k})!}. \end{aligned}$$

This completes the proof.  $\square$

**3.3. Proof of the main theorem.** We start with the following lemma.

**Lemma 3.7.** *For  $\mathbf{k} \in \mathbb{Z}_{\geq 1}^s$ , we have*

$$(3.1) \quad F_{\mathbf{k}}(t) = \sum_{\mathbf{w} \in W_{s-1}} F_{r_{\mathbf{w}}(\mathbf{k}) * \tilde{\mathbf{w}}}(t) \cdot t^{|\mathbf{w}|}.$$

*Proof.* Note that the left hand side of (3.1) is equal to

$$F_{\mathbf{k}}(t) = \sum_{\gamma \in \mathcal{F}(\Gamma_{\mathbf{k}})} t^{\dim \gamma}$$

by definition, and the right hand side of (3.1) is equal to

$$\sum_{\mathbf{w} \in W_{s-1}} F_{r_{\mathbf{w}}(\mathbf{k}) * \tilde{\mathbf{w}}}(t) \cdot t^{|\mathbf{w}|} = \sum_{\substack{\mathbf{w} \in W_{s-1} \\ \sigma \in \mathcal{F}(\Gamma_{r_{\mathbf{w}}(\mathbf{k}) * \tilde{\mathbf{w}}})}} t^{\dim \sigma} \cdot t^{|\mathbf{w}|}.$$

Let

$$X = \{(\mathbf{w}, \sigma) : \mathbf{w} \in W_{s-1}, \sigma \in \mathcal{F}(\Gamma_{r_{\mathbf{w}}(\mathbf{k}) * \tilde{\mathbf{w}}})\}.$$

Then it is sufficient to find a bijection

$$\phi : X \rightarrow \mathcal{F}(\Gamma_{\mathbf{k}})$$

such that  $\dim \phi(\mathbf{w}, \sigma) = |\mathbf{w}| + \dim \sigma$ .

Recall that the set of terminal vertices of  $\sigma \in \mathcal{F}(\Gamma_{r_{\mathbf{w}}(\mathbf{k}) * \tilde{\mathbf{w}}})$  is equal to  $V_{n-1}(g_{\mathbf{w}})$  where

$$V_{n-1}(g_{\mathbf{w}}) = V(g_{\mathbf{w}}) \cap \{(x, y) \mid x + y = n - 1\}.$$

Thus we can define  $\phi$  to be

$$\phi(\mathbf{w}, \sigma) := \sigma \cup g_{\mathbf{w}}$$

which is clearly a face of  $\Gamma_{\mathbf{k}}$ .

Conversely, any face  $\gamma \in \mathcal{F}(\Gamma_{\mathbf{k}})$  contains  $g_{\mathbf{w}}$  where  $\mathbf{w}$  corresponds to the assignment  $\mathcal{A}_{\gamma}$  and it can be decomposed into

$$\gamma = \sigma \cup g_{\mathbf{w}}$$

where  $\sigma \in \mathcal{F}(\Gamma_{r_{\mathbf{w}}(\mathbf{k}) * \tilde{\mathbf{w}}})$  is a full subgraph of  $\gamma$  obtained from removing terminal vertices  $T_{\mathbf{k}}$  of  $\gamma$ . Then it defines a map  $\psi : \mathcal{F}(\Gamma_{\mathbf{k}}) \rightarrow X$  such that  $\psi(\gamma) := (\sigma, \mathbf{w})$ . It is clear that  $\psi \circ \phi$  is the identity map on  $X$ .

Finally for every  $(\mathbf{w}, \sigma) \in X$ , each  $v_i$  with  $(\alpha_i, \beta_i) = (1, 1)$  generates exactly one cycle in  $\phi(\mathbf{w}, \sigma)$  containing  $v_i$ . Thus we have  $\dim \phi(\mathbf{w}, \sigma) = |\mathbf{w}| + \dim \sigma$ . □

Note that

$$\frac{\partial^s}{\partial x_1 \cdots \partial x_s} \Psi_s(\mathbf{x}; t) = \frac{\partial^s}{\partial x_1 \cdots \partial x_s} \Psi_{2s-1}(\mathbf{x} * \mathbf{y}; t) \Big|_{\mathbf{y}=0}.$$

Thus the following theorem is equivalent to our main theorem 1.12.

**Theorem 3.8** (Theorem 1.12). *Under the same assumption,*

$$\frac{\partial^s}{\partial x_1 \cdots \partial x_s} \Psi_s(\mathbf{x}; t) = \left( \prod_{i=1}^{s-1} \left( \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_{i+1}} + t \cdot \frac{\partial}{\partial y_i} \right) \right) \Big|_{\mathbf{y}=0} \Psi_{2s-1}(\mathbf{x} * \mathbf{y}; t).$$

*Proof.* By Lemma 3.7, the left hand side is

$$(3.2) \quad \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 1}^s} F_{\mathbf{k}}(t) \frac{\mathbf{x}^{\mathbf{k}-1}}{(\mathbf{k}-1)!} = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^s} F_{\mathbf{k}+1}(t) \frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!} = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^s} \left( \sum_{\mathbf{w} \in W_{s-1}} F_{r_{\mathbf{w}}(\mathbf{k}+1) * \tilde{\mathbf{w}}}(t) \cdot t^{|\mathbf{w}|} \right) \frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}!}.$$

Observe that

$$\Psi_{2s-1}(\mathbf{x} * \mathbf{y}; t) = \sum_{\substack{\mathbf{k} \in \mathbb{Z}_{\geq 0}^s \\ \mathbf{e} \in \mathbb{Z}_{\geq 0}^{s-1}}} F_{\mathbf{k} * \mathbf{e}}(t) \frac{(\mathbf{x} * \mathbf{y})^{\mathbf{k} * \mathbf{e}}}{(\mathbf{k} * \mathbf{e})!}.$$

By Lemma 3.5 and the above identity, the right hand side of the theorem is

$$\left( \sum_{\mathbf{w} \in W_{s-1}} D_{\mathbf{w}} (\Psi_{2s-1}(\mathbf{x} * \mathbf{y})) \right) \Big|_{\mathbf{y}=0} = \sum_{\mathbf{w} \in W_{s-1}} \sum_{\substack{\mathbf{k} \in \mathbb{Z}_{\geq 0}^s \\ \mathbf{e} \in \mathbb{Z}_{\geq 0}^{s-1}}} F_{\mathbf{k} * \mathbf{e}}(t) \left( D_{\mathbf{w}} \left( \frac{(\mathbf{x} * \mathbf{y})^{\mathbf{k} * \mathbf{e}}}{(\mathbf{k} * \mathbf{e})!} \right) \right) \Big|_{\mathbf{y}=0}.$$

By Lemma 3.6, this is equal to

$$\sum_{\mathbf{w} \in W_{s-1}} \sum_{\substack{\mathbf{k} \in \mathbb{Z}_{\geq 0}^s \\ d_{\mathbf{w}}(\mathbf{k}) \in \mathbb{Z}_{\geq 0}^s}} F_{\mathbf{k} * \tilde{\mathbf{w}}}(t) \cdot t^{|\mathbf{w}|} \cdot \frac{\mathbf{x}^{d_{\mathbf{w}}(\mathbf{k})}}{d_{\mathbf{w}}(\mathbf{k})!}.$$

Note that  $d_{\mathbf{w}}(\mathbf{k}) \in \mathbb{Z}_{\geq 0}^s$  implies  $\mathbf{k} = r_{\mathbf{w}}(d_{\mathbf{w}}(\mathbf{k}) + \mathbf{1}) \in Z_{\geq 0}^s$  by Lemma 3.4. Thus by letting  $\mathbf{k}' = d_{\mathbf{w}}(\mathbf{k})$ , the above sum becomes

$$\sum_{\mathbf{w} \in W_{s-1}} \sum_{\mathbf{k}' \in \mathbb{Z}_{\geq 0}^s} F_{r_{\mathbf{w}}(\mathbf{k}'+1) * \tilde{\mathbf{w}}}(t) \cdot t^{|\mathbf{w}|} \cdot \frac{\mathbf{x}^{\mathbf{k}'}}{(\mathbf{k}')!},$$

which is equal to (3.2). This completes the proof.  $\square$

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CENTER FOR GEOMETRY AND PHYSICS, INSTITUTE FOR BASIC SCIENCE (IBS), POHANG, REPUBLIC OF KOREA 37673  
*E-mail address*: anbyhee@ibs.re.kr

CENTER FOR GEOMETRY AND PHYSICS, INSTITUTE FOR BASIC SCIENCE (IBS), POHANG, REPUBLIC OF KOREA 37673  
*E-mail address*: yhcho@ibs.re.kr

DEPARTMENT OF MATHEMATICS, SUNGKYUNKWAN UNIVERSITY, SUWON 440-746, SOUTH KOREA  
*E-mail address*: jangsookim@skku.edu